CS3911 Intro. to Numerical Methods with Fortran Exam 2 Solutions
Fall 2007

1. **Systems of Linear Equations**

   (a) [**6 points**] Answer the following **two** questions: **(1)** define the meaning of a *diagonal dominant* matrix $A = [a_{i,j}]_{n\times n}$; and **(2)** what is the advantage(s) for a matrix to be diagonal dominant?

   Answer: A matrix $A = [a_{i,j}]_{n\times n}$ is said to be *diagonal dominant* if the following holds for every row:

$$|a_{i,i}| > \sum_{j=1, j\neq i}^{n} |a_{i.j}| \qquad \text{for } i = 1, 2, \ldots, n$$

   This means for each row the magnitude (*i.e.*, absolute value) of the diagonal entry is greater than the sum of magnitude of other entries on the same row.

   For diagonal dominant matrices, Gaussian elimination and LU-decomposition does not require partial pivoting, and the Jacobi and Gauss-Seidel iterative methods converge. ■

   (b) [**20 points**] Let ϵ be a very small positive number (*i.e.*, $\epsilon \approx 0$). Do the following **two** problems: **(1)** solve the following system of linear equations *with* **and** *without* partial pivoting; and **(2)** find the major reason or reasons that can explain the difference(s) between the two solutions. **You have to clearly state your findings with a convincing argument. Just stating a "reason" such as "it is because of cancelation" or "overflow" will receive zero point.**

$$\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   Answer: Without pivoting, one multiplies $-1/\epsilon$ to the first equation and adds the result to the second. This yields the following:

$$\begin{bmatrix} \epsilon & 1 \\ 0 & 1-1/\epsilon \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2-1/\epsilon \end{bmatrix}$$

   Since ϵ is small, $1/\epsilon$ is large. As a result, $-1/\epsilon$ is the dominating term in both $1 - 1/\epsilon$ and $2 - 1/\epsilon$. In other word, the equation $(1 - 1/\epsilon)y = 2 - \epsilon$ would numerically become

$$\left(-\frac{1}{\epsilon}\right) y \approx -\frac{1}{\epsilon}$$

   Therefore, $y = 1$. Plugging $y = 1$ into the first equation $\epsilon x + y = 1$ yields $x = 0$.

   With pivoting, the system becomes the following:

$$\begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   Multiplying the first equation by $-\epsilon$ and adding the result to the second yields:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1-\epsilon \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1-2\epsilon \end{bmatrix}$$

   Since ϵ is very small, it does not contribute much to $1 - \epsilon$ and $1 - 2\epsilon$. As a result, $1 - \epsilon \approx 1$ and $1 - 2\epsilon \approx 1$, and the second equation numerically becomes $y = 1$. Plugging $y = 1$ into the first equation gives $x = 1$.

Obviously, $x = y = 1$ is the correct numerical solution if ϵ is small. The reason for the non-pivoting solution to go wrong is the rounding error in computing $1 - 1/\epsilon$ and $2 - 1/\epsilon$ if $1/\epsilon$ is very large. In this case, rounding error makes both terms nearly equal to $-1/\epsilon$ numerically. Consequently, we have $y = 1$ which is still correct. But, since ϵ is very small, $x = 1 - \epsilon$ will have rounding error again. This time, the impact of ϵ is so small that becomes insignificant compared with 1. Therefore, $x = 0$!
For example, on a 7-digit computer, if $\epsilon = 0.00000001$ we have $1/\epsilon = 100,000,000$. Then, $1 - 1/\epsilon = -99,999,999$ and $2 - 1/\epsilon = -99,999,998$. Both would be rounded to 7-digit and the result is 0.1×10^9. This example shows that pivoting is necessary. ■

(c) **[15 points]** Given matrix A as shown below, find its LU-decomposition *without* pivoting. **You have to show <u>all</u> computation steps, and explain how you get the results. Otherwise (*e.g.*, providing an answer only and/or asking me to guess your intention from a bunch of numbers), you will receive <u>zero</u> point.**

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

<u>Answer</u>: A lower triangular matrix L has the following form, where × indicates a value to be determined in the process of Gaussian elimination.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & \times & 1 \end{bmatrix}$$

Multiplying 0 and -2 to the first row of A and adding the results to the second and third rows, respectively, we have L and U as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & \times & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Note that the multipliers (*i.e.*, 0 and -2) are saved to the first column of L with opposite signs.
Then, multiplying -1 to the second row (of U) and adding the result to the third yields:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify $A = L \cdot U$ yourself. ■

(d) **[12 points]** Use Gauss-Seidel method to solve the following system of linear equations, and fill the table below with your results. The initial value (*i.e.*, iteration 0) is $x = y = z = 0$, and you only do two iterations (*i.e.*, iterations 1 and 2).

$$\begin{array}{rcrcrcr} 4x & + & y & + & z & = & 1 \\ x & + & 4y & + & z & = & 2 \\ x & + & y & +4 & z & = & 4 \end{array}$$

Iteration	x	y	z
0	0	0	0
1	$\frac{1}{4} = 0.25$	$\frac{7}{16} = 0.4375$	$\frac{53}{64} = 0.828125$
2	$-\frac{17}{256} = -0.06640625$	$\frac{317}{1024} = 0.309570312$	$\frac{3847}{4096} = 0.939208984$

(e) **[20 points]** Suppose a program read in the following system of linear equations:

$$A \cdot x = B \qquad \text{where } A = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and delivered the solution $x = y = 1$. However, this solution is obviously inaccurate. Suppose we happen to know the LU-decomposition of A as shown below. Use the iterative refinement method to improve the accuracy of this "solution." **You have to show <u>all</u> computation steps, and explain how you get the results. Otherwise (*e.g.*, only providing an answer and/or asking me to guess your intention from a bunch of numbers), you will receive <u>zero</u> point.**

$$A = L \cdot U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

<u>Answer</u>: The following shows all computation steps:

- **<u>Compute the error vector r</u>**:

$$r = B - A \cdot X = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

- **<u>Forward substitution to find T in $L \cdot T = r$</u>**:
  The equation of $L \cdot T = r$ is the following:

$$\begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

  Forward substitution gives $t_1 = -2$. Plugging $t_1 = -2$ into $t_1 + t_2 = -3$ yields $t_2 = -1$.

- **<u>Backward substitution to find Δ in $U \cdot \Delta = T$</u>**:
  The equation of $U \cdot \Delta = T$ is shown below:

$$\begin{bmatrix} 1 & 3 \\ & 2 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

  Backward substitution gives $\delta_2 = -1/2$. Plugging $\delta_2 = -1/2$ into the first equation $\delta_1 + 3\delta_2 = -2$ yields $\delta_1 = -1/2$. Therefore, $\Delta = [\delta_1, \delta_2]^T = [-0.5, -0.5]^T$.

- **<u>Compute the new X</u>**:
  The new X is computed as $X + \Delta$: new$X = [1, 1]^T + [-0.5, -0.5]^T = [0.5, 0.5]^T$.

- **<u>Verify the computed result</u>**:
  Since $B - A \cdot [0.5, 0.5]^T = [0, 0]^T$, we have computed the correct solution to the system of linear equations. ∎

2. **Eigenvalues and Eigenvectors**

   (a) **[12 points]** Use the power method to find the largest eigenvalue and its corresponding eigenvector of matrix A as shown below, and fill the following table with your results. The initial value (*i.e.*, iteration 0) is $z = [1, 1]^T$, and you only do two iterations (*i.e.*, iterations 2 and 3).

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Iteration	*Approx.*	*Approx. Eigenvector*	
	Eigenvalue	x	y
0	1	1	1
1	7	$\frac{7}{3} = 0.428571428$	1
2	$\frac{33}{7} = 4.714285714$	$\frac{17}{33} = 0.515151515$	1

   (b) **[15 points]** Use Jacobi method to find *all* eigenvalues and their corresponding eigenvectors of the following symmetric matrix A. **You should provide clearly <u>all</u> computation details, and match each eigenvalue with its corresponding eigenvector. Otherwise, you will risk low grade. Additionally, you will receive <u>zero</u> point if you do not use Jacobi method.**

$$A = \begin{bmatrix} 2 & -2\sqrt{3} \\ -2\sqrt{3} & 6 \end{bmatrix}$$

<u>Answer</u>: The Jacobi method first determines a rotation angle θ, from which a rotation matrix is constructed. Since the only non-zero off-diagonal element is $a_{1,2}$, we have

$$\tan(2\theta) = \frac{2a_{1,2}}{a_{2,2} - a_{1,1}} = \frac{2 \times (-2\sqrt{3})}{6-2} = -\sqrt{3}$$

Therefore, $2\theta = -\pi/3$ and $\theta = -\pi/6$, and the rotation matrix R is

$$R = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

The rotated matrix A is

$$A' = R^T \cdot A \cdot R = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}$$

and the eigenvector matrix V is

$$V = I \cdot R = R = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Hence, A's eigenvalues are 0 and 8 with corresponding eigenvectors $[\sqrt{3}/2, 1/2]^T$ and $[-1/2, \sqrt{3}/2]^T$. ■