Constraint Satisfaction Problems

Chapter 5

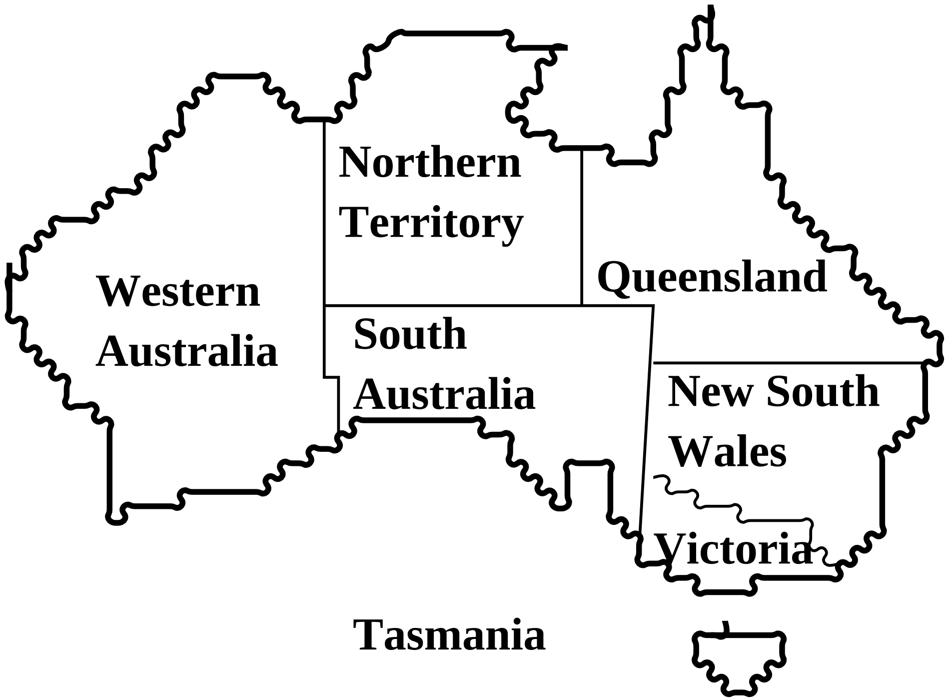
Outline

- CSP examples
- Backtracking search for CSPs
- Problem structure and problem decomposition
- Local search for CSPs

Constraint satisfaction problems (CSPs)

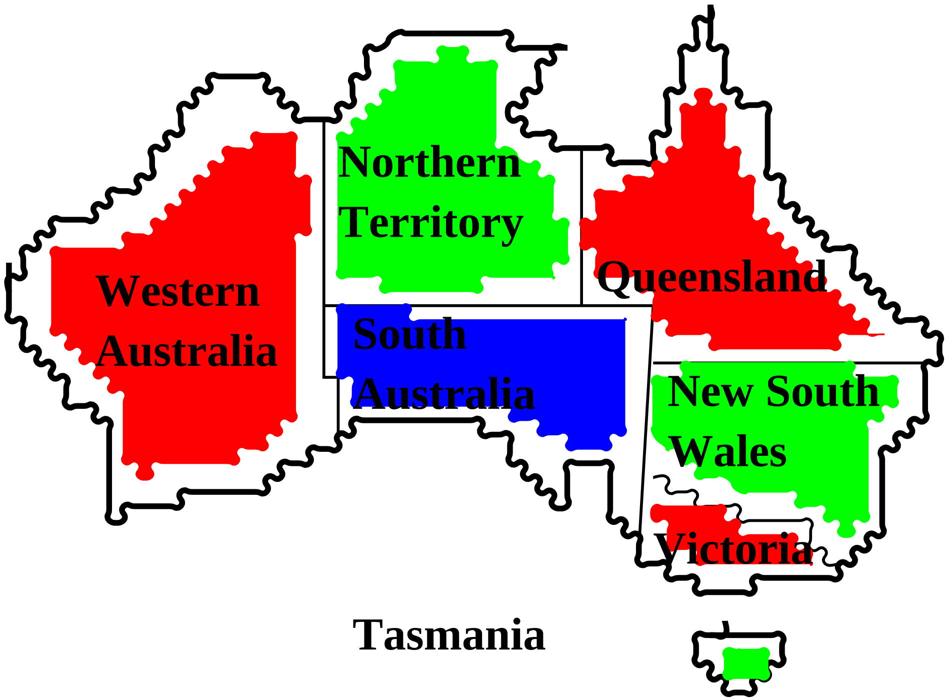
- Standard search problem:
  state is a "black box"—any old data structure that supports goal test, eval, successor
- CSP:
  state is defined by *variables* X_i
  with *values* from *domain* D_i
  goal test is a set of *constraints* specifying allowable combinations of values for subsets of variables
- Simple example of a *formal representation language*
- Allows useful *general-purpose* algorithms with more power than standard search algorithms

Example: Map-Coloring



Variables WA, NT, Q, NSW, V, SA, T
Domains $D_i = \{red, green, blue\}$
Constraints: adjacent regions must have different colors
e.g., $WA \neq NT$ (if the language allows this), or
$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

Example: Map-Coloring (cont'd)

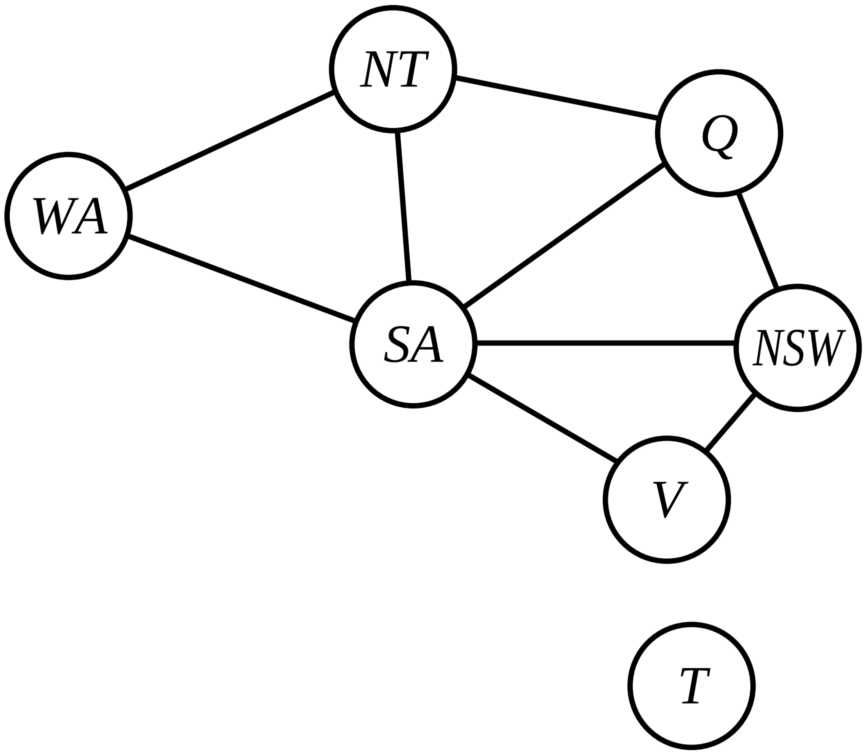


Solutions are assignments satisfying all constraints, e.g.,

$\{WA = red, NT = green, Q = red, NSW = green,$
$V = red, SA = blue, T = green\}$

Constraint graph

- *Binary CSP*: each constraint relates at most two variables
- *Constraint graph*: nodes are variables, arcs show constraints



- General-purpose CSP algorithms use the graph structure to speed up search.
  E.g., Tasmania is an independent subproblem!

Varieties of CSPs

- Discrete variables
  - finite domains; size $d \implies O(d^n)$ complete assignments
    e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  - infinite domains (integers, strings, etc.)
    e.g., job scheduling, variables are start/end days for each job
    need a *constraint language*, e.g.,
    $StartJob_1 + 5 \leq StartJob_3$
  - linear constraints solvable, nonlinear undecidable
- Continuous variables
  e.g., start/end times for Hubble Telescope observations

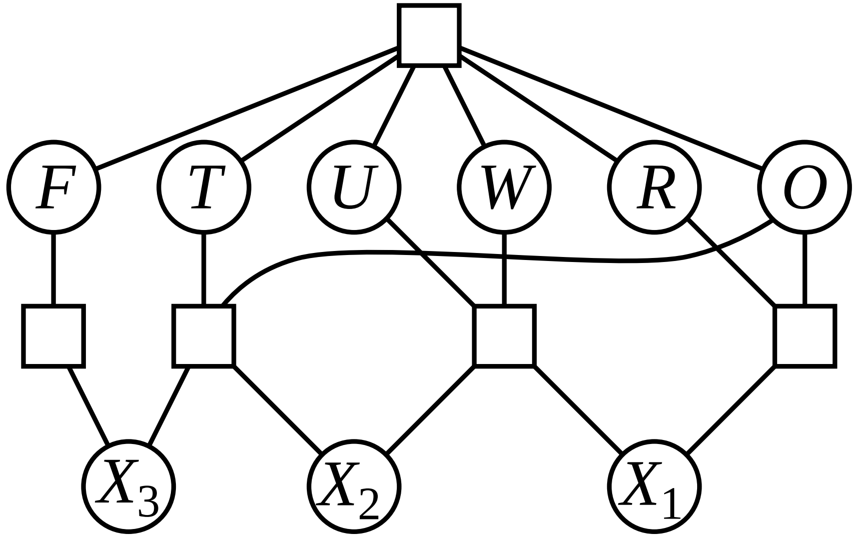
Varieties of constraints

- *Unary* constraints involve a single variable, e.g., $SA \neq green$
- *Binary* constraints involve pairs of variables, e.g., $SA \neq WA$
- *Higher-order* constraints involve 3 or more variables, e.g., cryptarithmetic column constraints
- *Preferences* (soft constraints), e.g., red is better than $green$
  often representable by a cost for each variable assignment
  $\rightarrow$ constrained optimization problems

Example: Cryptarithmetic

$$\begin{array}{r} T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$

(a)



(b)

Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints

alldiff(F, T, U, W, R, O)

$O + O = R + 10 \cdot X_1$, etc.

Real-world CSPs

- Assignment problems
  e.g., who teaches what class
- Timetabling problems
  e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Floorplanning

Notice that many real-world problems involve real-valued variables

Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far

- Initial state: the empty assignment, $\emptyset$
- Successor function: assign a value to an unassigned variable
  that does not conflict with current assignment.
  $\Longrightarrow$ fail if no legal assignments (not fixable!)
- Goal test: the current assignment is complete

Standard search formulation (incremental)

- This is the same for all CSPs!
- Every solution appears at depth n with n variables $\implies$ use depth-first search
- Path is irrelevant, so can also use complete-state formulation
- $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!!

Backtracking search

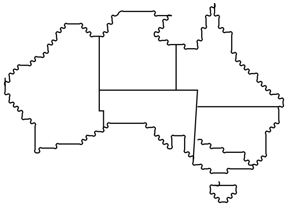
- Variable assignments are commutative, i.e.,
  $[WA = red$ then $NT = green]$ same as
  $[NT = green$ then $WA = red]$
- Only need to consider assignments to a single variable at each node
  $\Longrightarrow b = d$ and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments
  is called *backtracking* search
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for $n \approx 25$

Backtracking search

```
function BACKTRACKING-SEARCH (csp)
returns a solution, or failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING (assignment, csp)
returns a solution, or failure
  if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VAR(VARIABLES[csp], assignment,csp)
  for each value in ORDER-DOMAIN-VARS(var, assignment, csp) do
    add { var = value } to assignment
    result ← RECURSIVE-BACKTRACKING (assignment, csp)
    if result ≠ failure then return result
    remove { var = value } from assignment
return failure
```

Backtracking example



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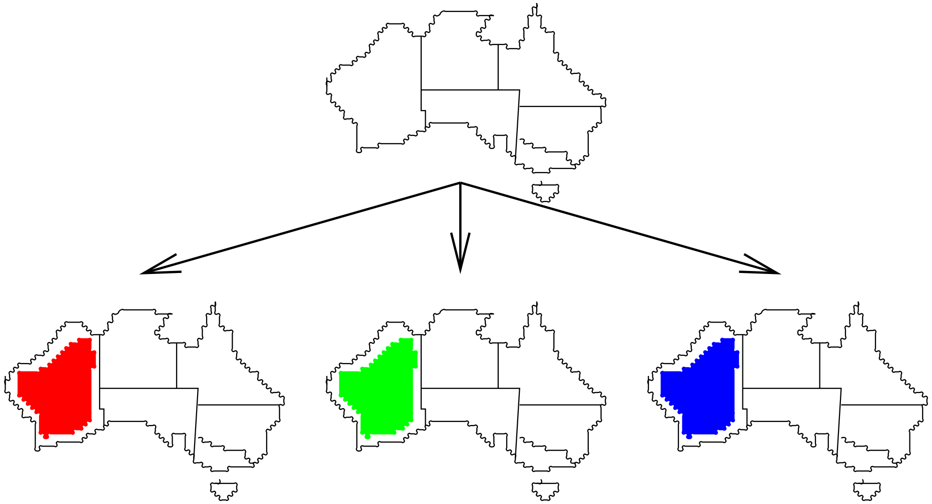

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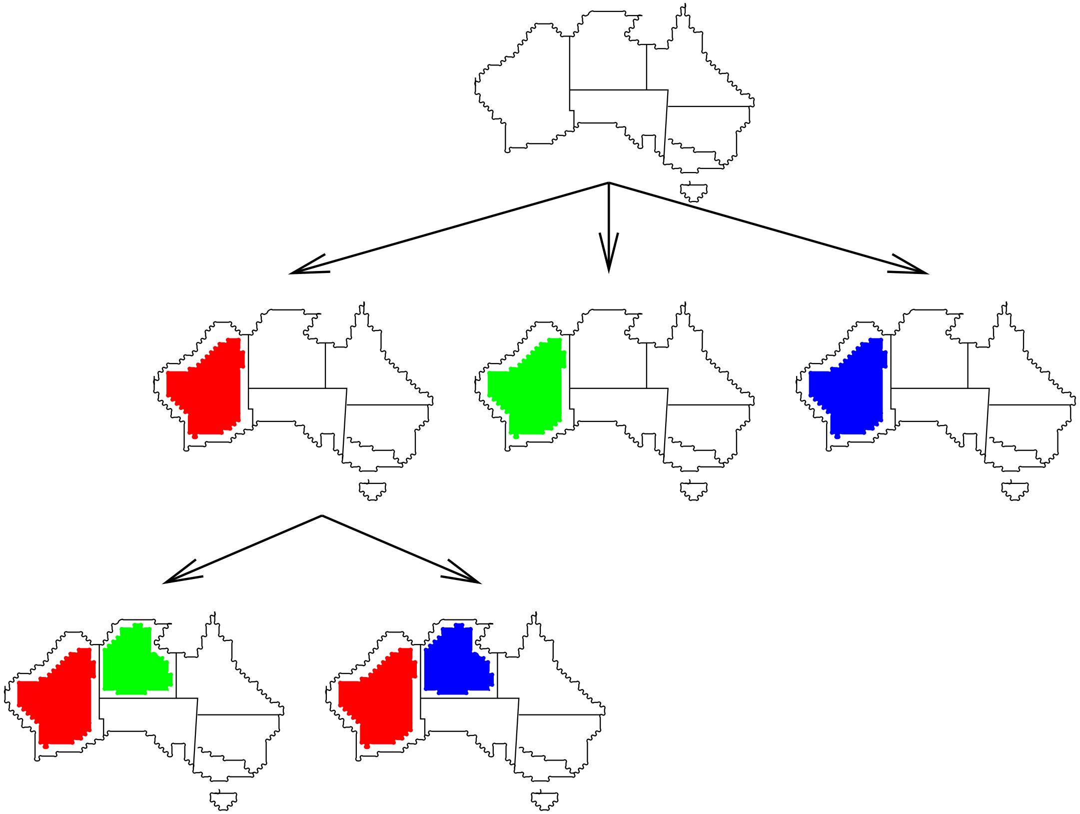

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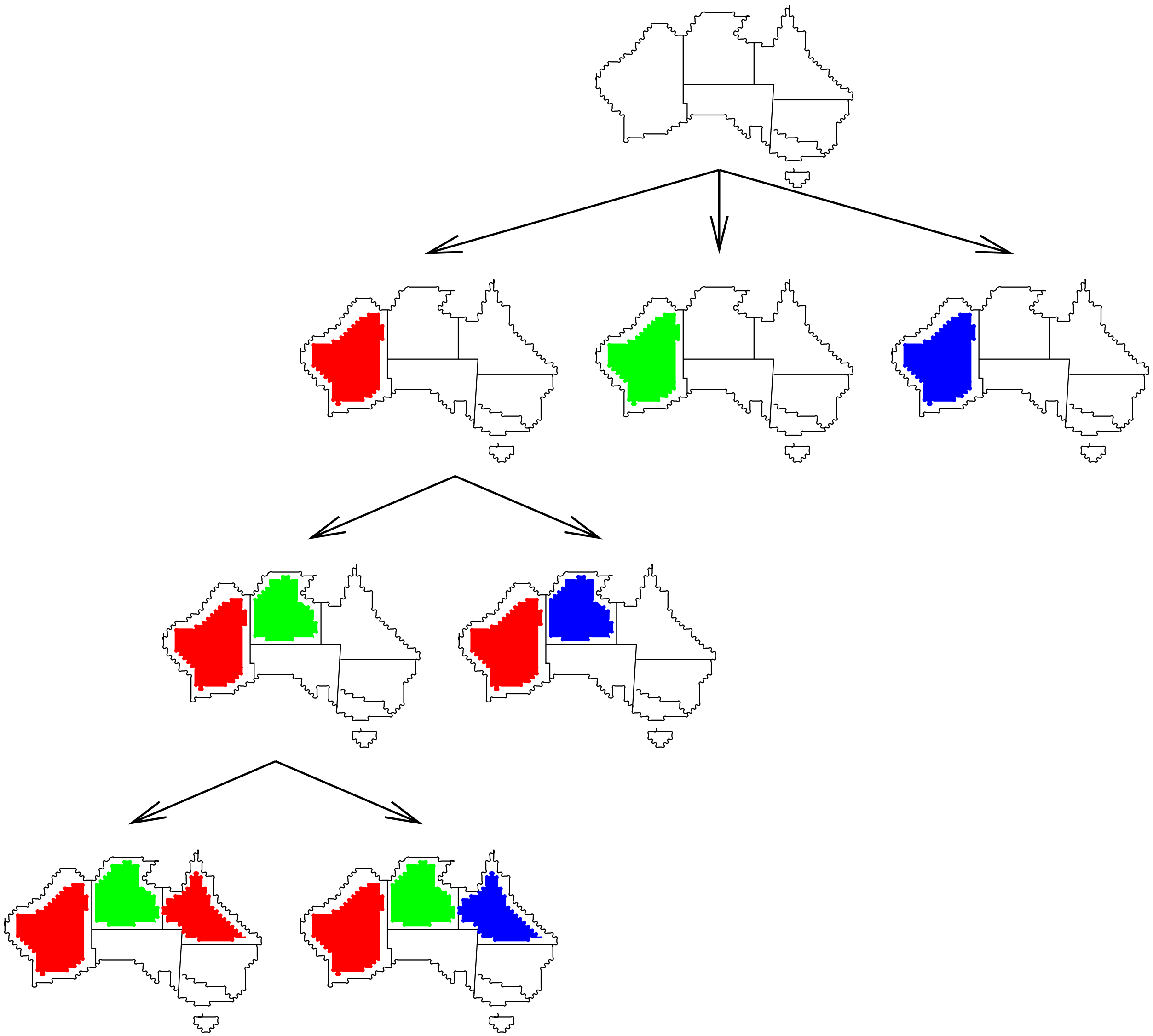


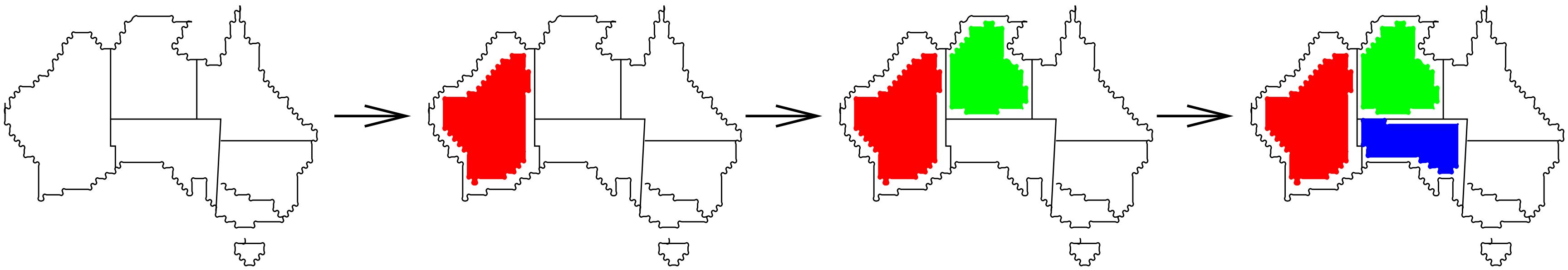
Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Most constrained variable

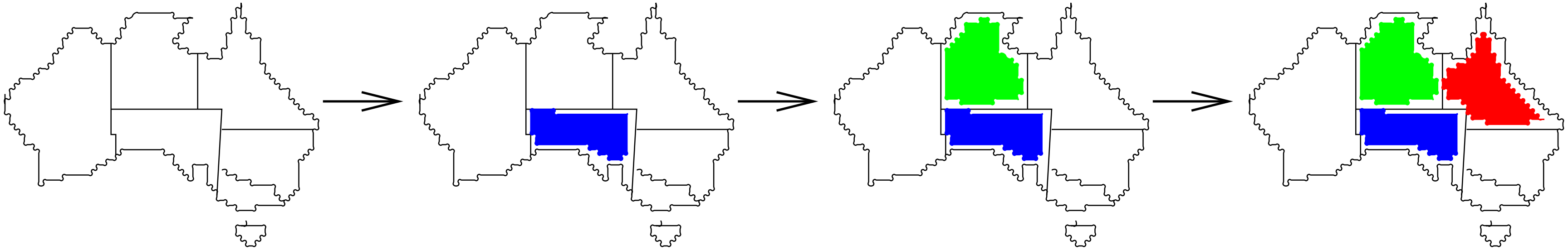
Most constrained variable:
choose the variable with the fewest legal values



Most constraining variable

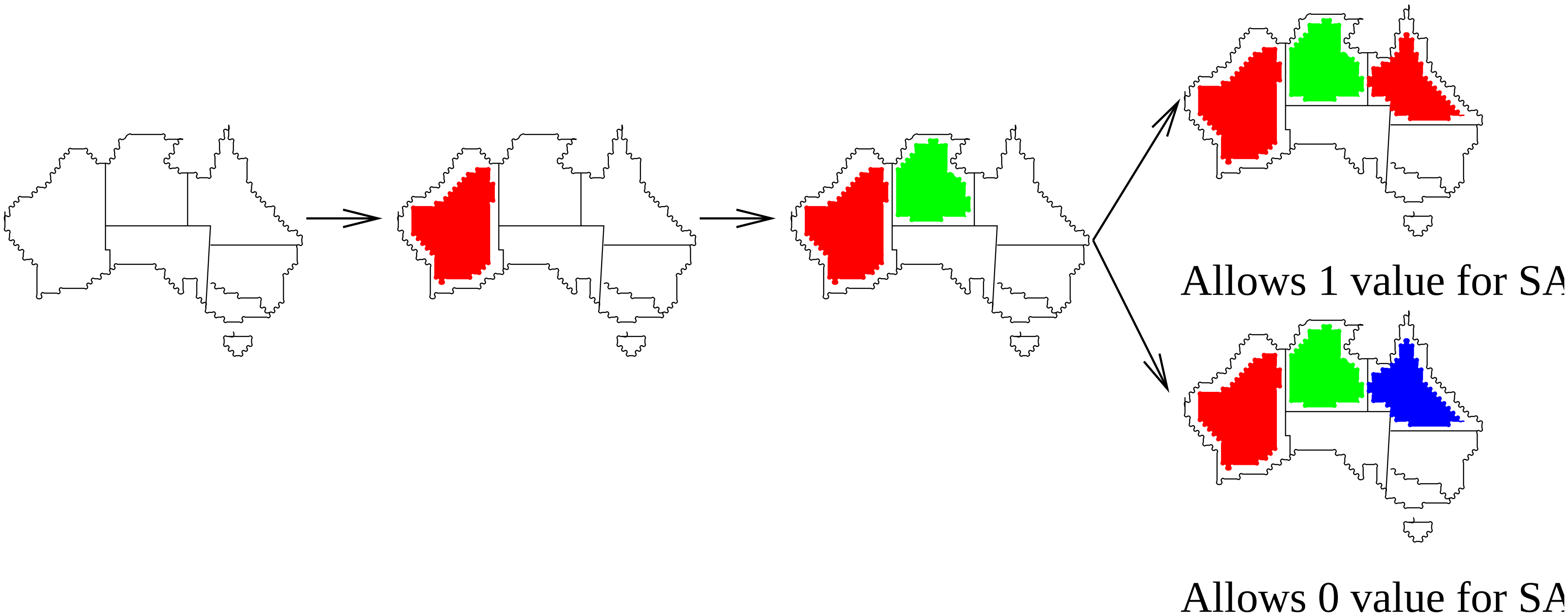
Tie-breaker among most constrained variables

Most constraining variable:
choose the variable with the most constraints on remaining variables



Least constraining value

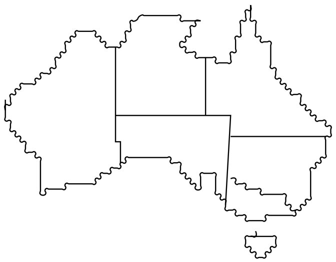
Given a variable, choose the least constraining value:
the one that rules out the fewest values in the remaining variables



Combining these heuristics makes 1000 queens feasible

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values



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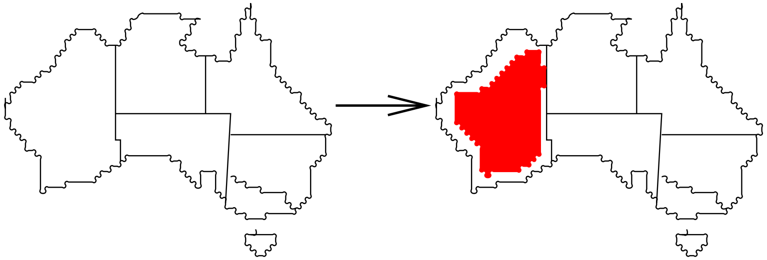

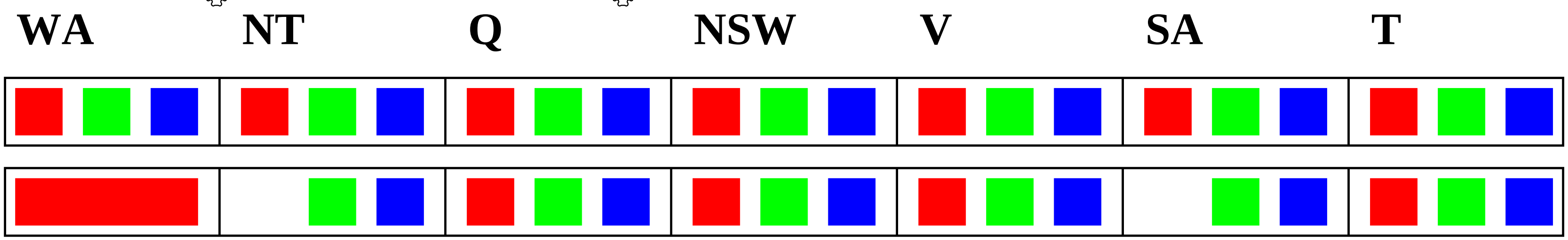

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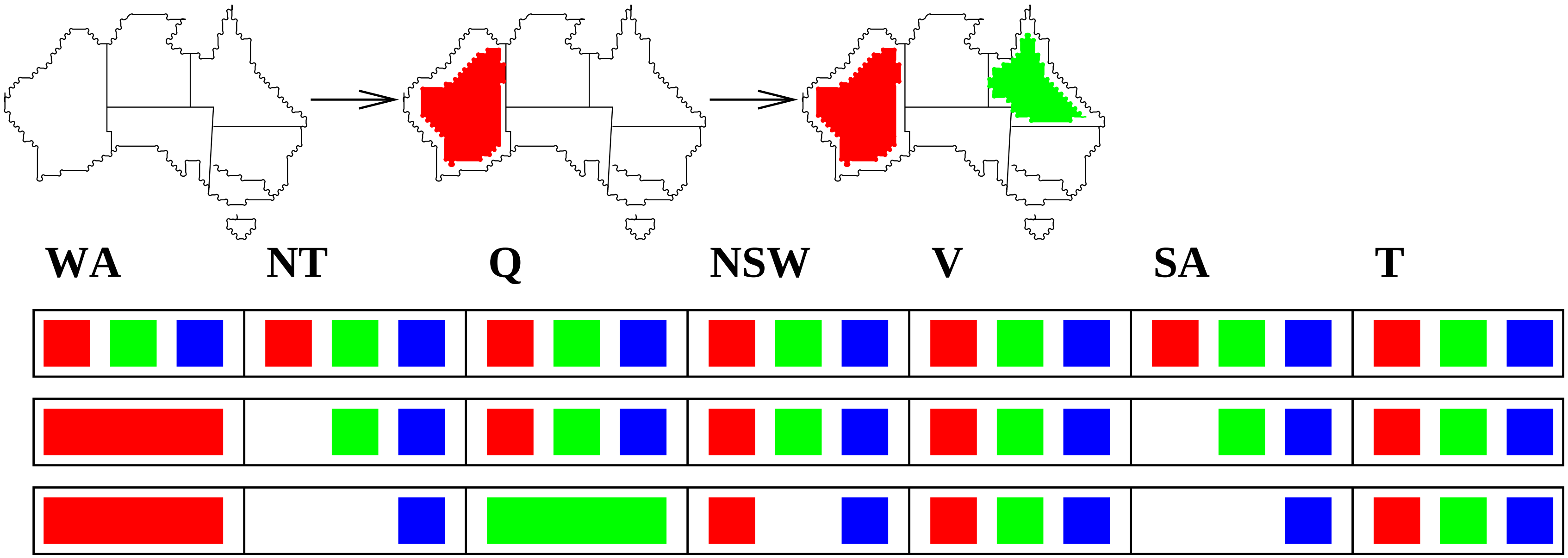

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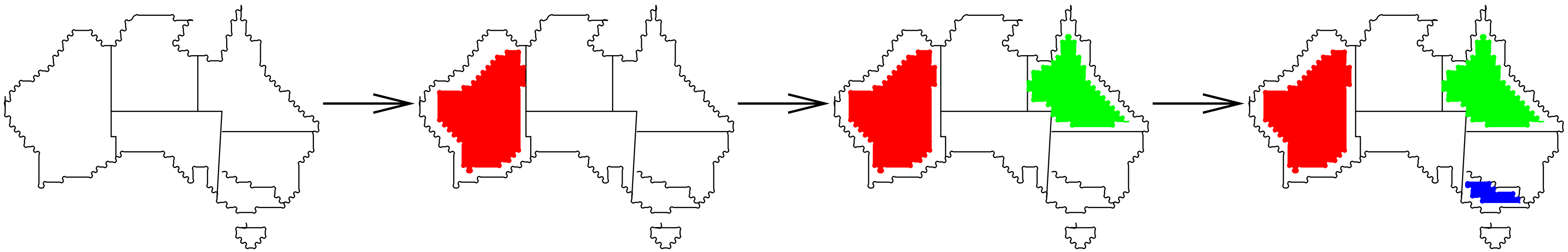
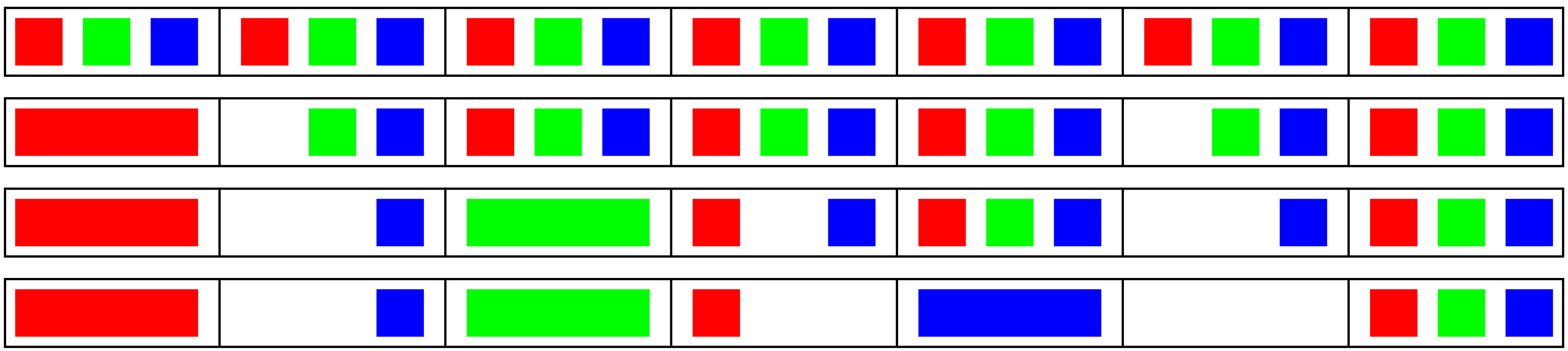
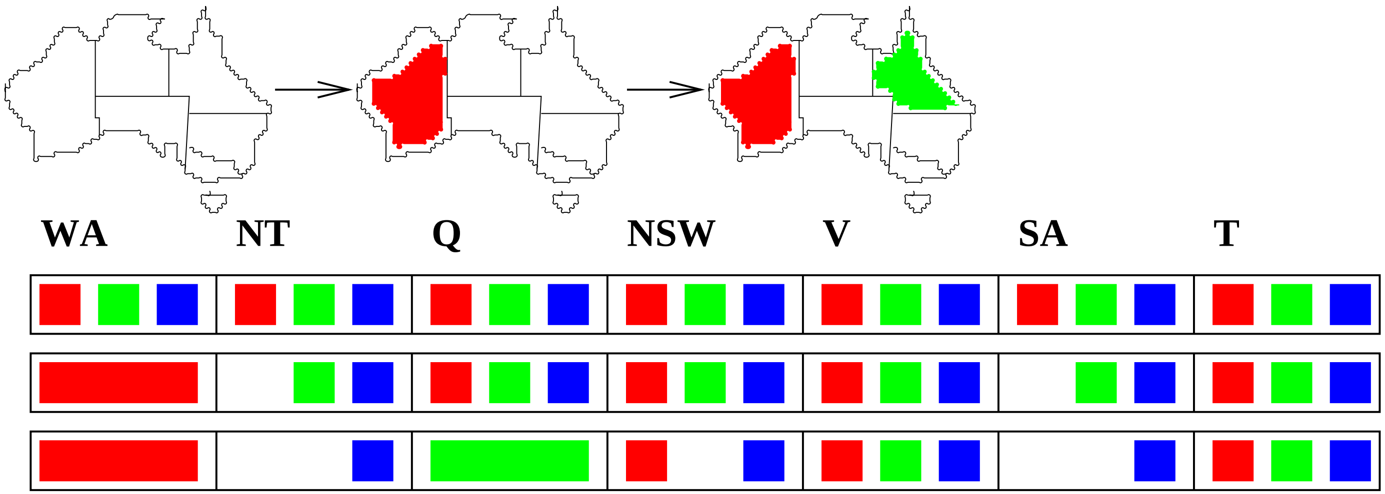


WA NT Q NSW V SA T



Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



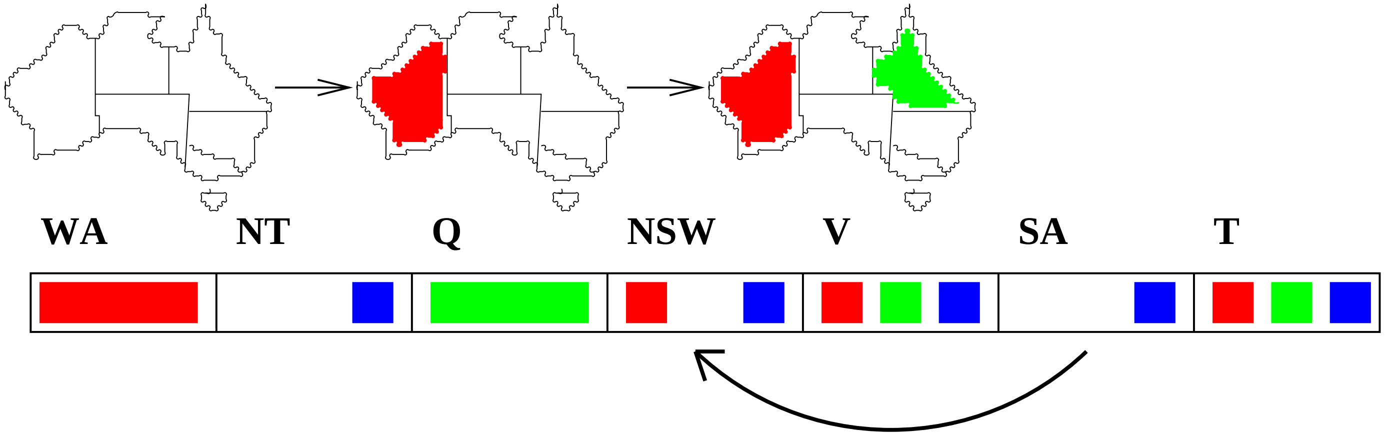
NT and SA cannot both be blue!

Constraint propagation repeatedly enforces constraints locally

Arc consistency

Simplest form of propagation makes each arc *consistent*

$X \rightarrow Y$ is consistent iff
for every value x of X there is some allowed y



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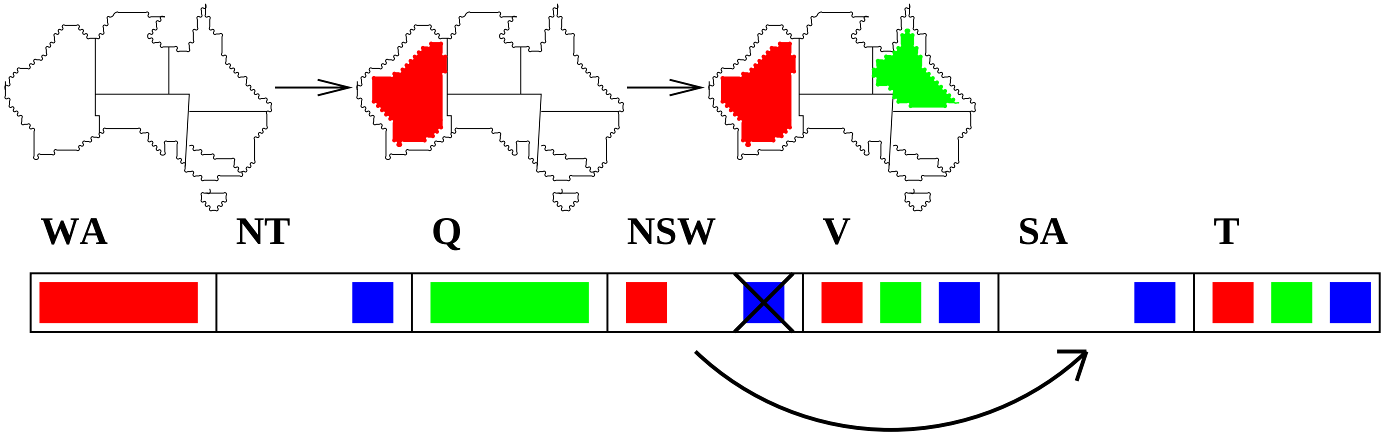

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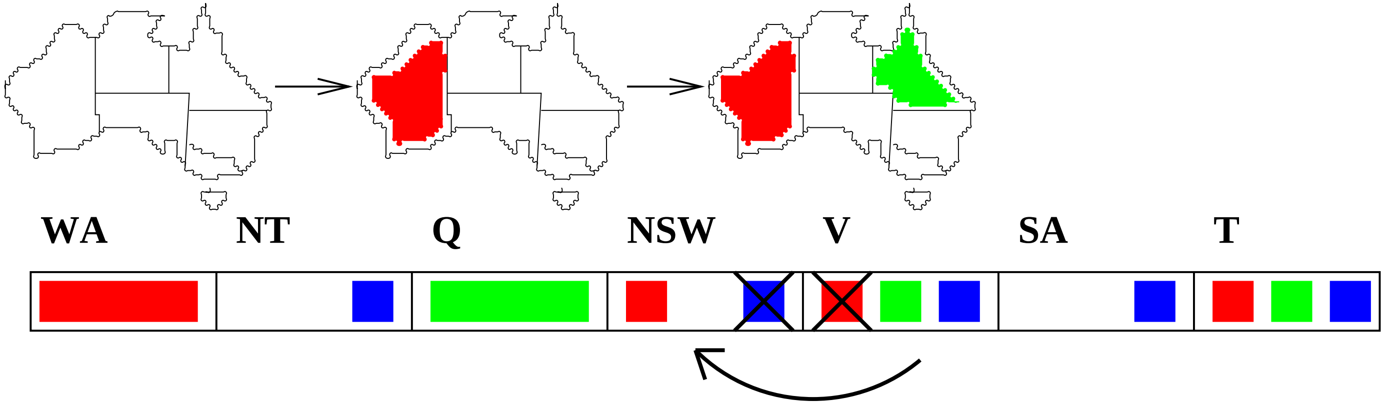

If X loses a value, neighbors of X need to be rechecked

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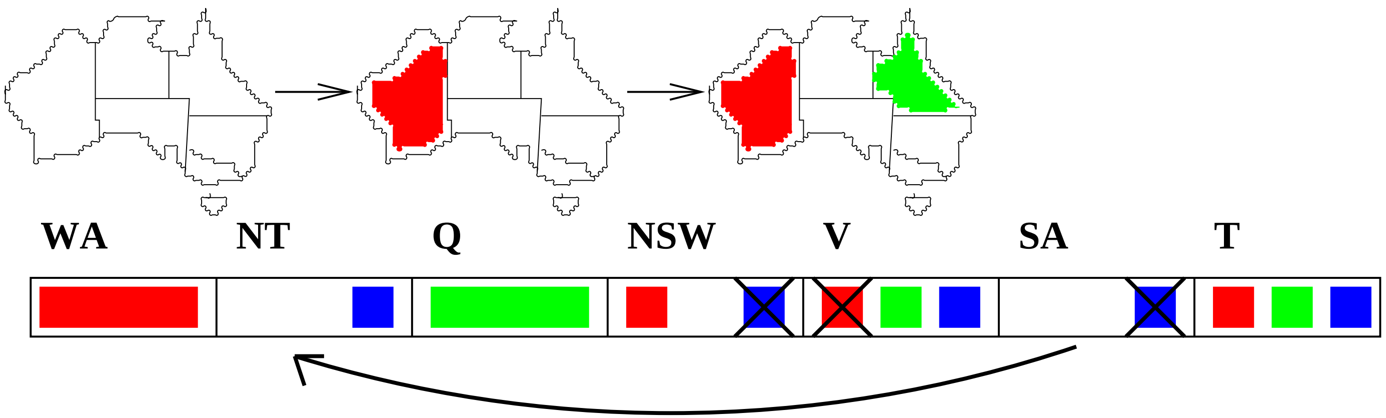


If X loses a value, neighbors of X need to be rechecked
Arc consistency detects failure earlier than forward checking
Can be run as a preprocessor or after each assignment

Arc consistency algorithm

```
function AC-3 (csp)
returns the CSP, with possibly reduced domains
   inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
   local variables: queue, a queue of arcs, initally all the arcs in csp

   while queue is not empty do
      (X_i, X_j) ← REMOVEFIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
         for each X_k in NEIGHBORS[X_i] do
            add (X_k, X_i) to queue
```

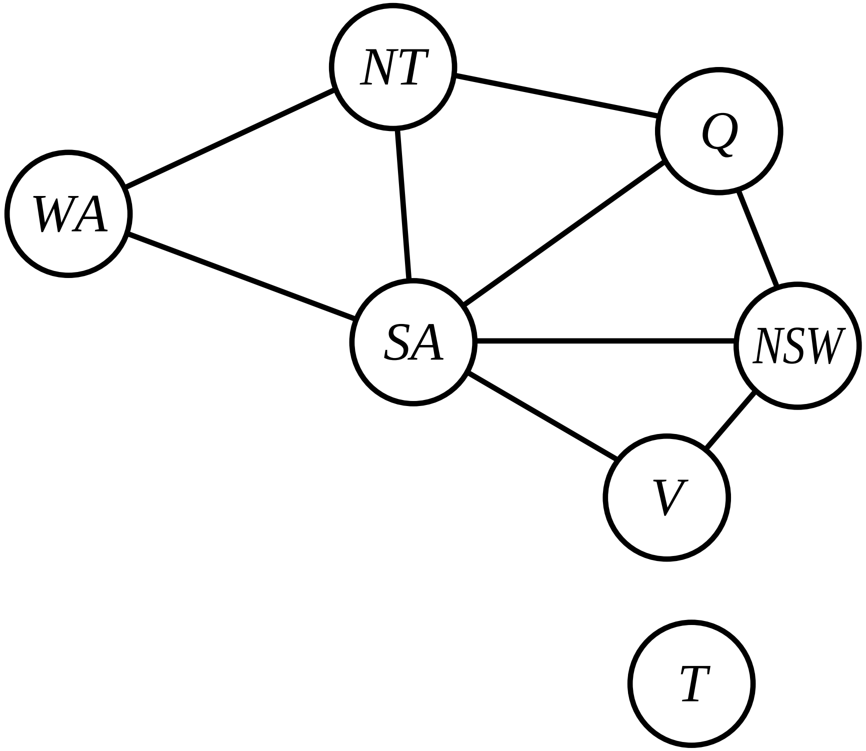
Arc consistency algorithm

function REMOVE-INCONSISTENT-VALUES (X_i, X_j)
returns true iff we remove a value

removed $\leftarrow$ false
for each *x* in DOMAIN$[X_i]$ **do**
  if no value in DOMAIN$[X_j]$ allows (x, y) to satisfy the constraint between X_i and X_j
    then delete x from DOMAIN$[X_i]$;
      removed $\leftarrow$ true
return *removed*

$O(n^2d^3)$, can be reduced to $O(n^2d^2)$
but cannot detect all failures in poly time!

Problem structure



Tasmania and mainland are *independent subproblems*
Identifiable as *connected components* of constraint graph

Problem structure contd.

Suppose each subproblem has c variables out of n total

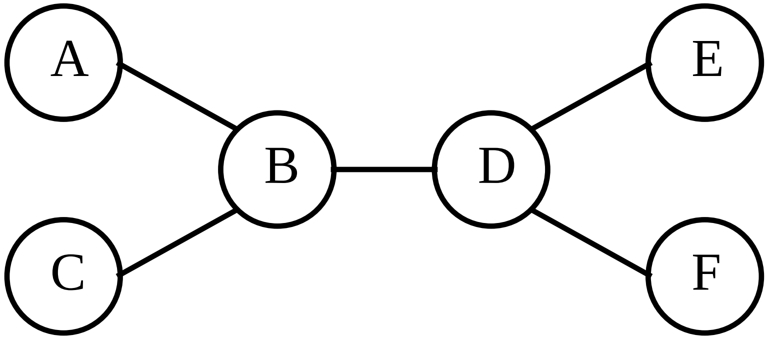
Worst-case solution cost is $n/c \cdot d^c$, linear in n

E.g., $n = 80$, $d = 2$, $c = 20$

2^{80} = 4 billion years at 10 million nodes/sec

$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

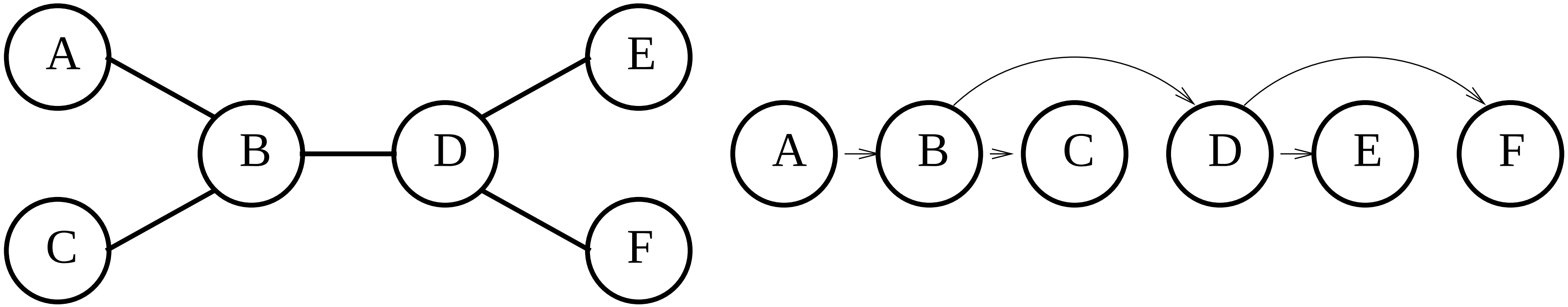
Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning:

an important example of the relation between syntactic restrictions and the complexity of reasoning.

Algorithm for tree-structured CSPs

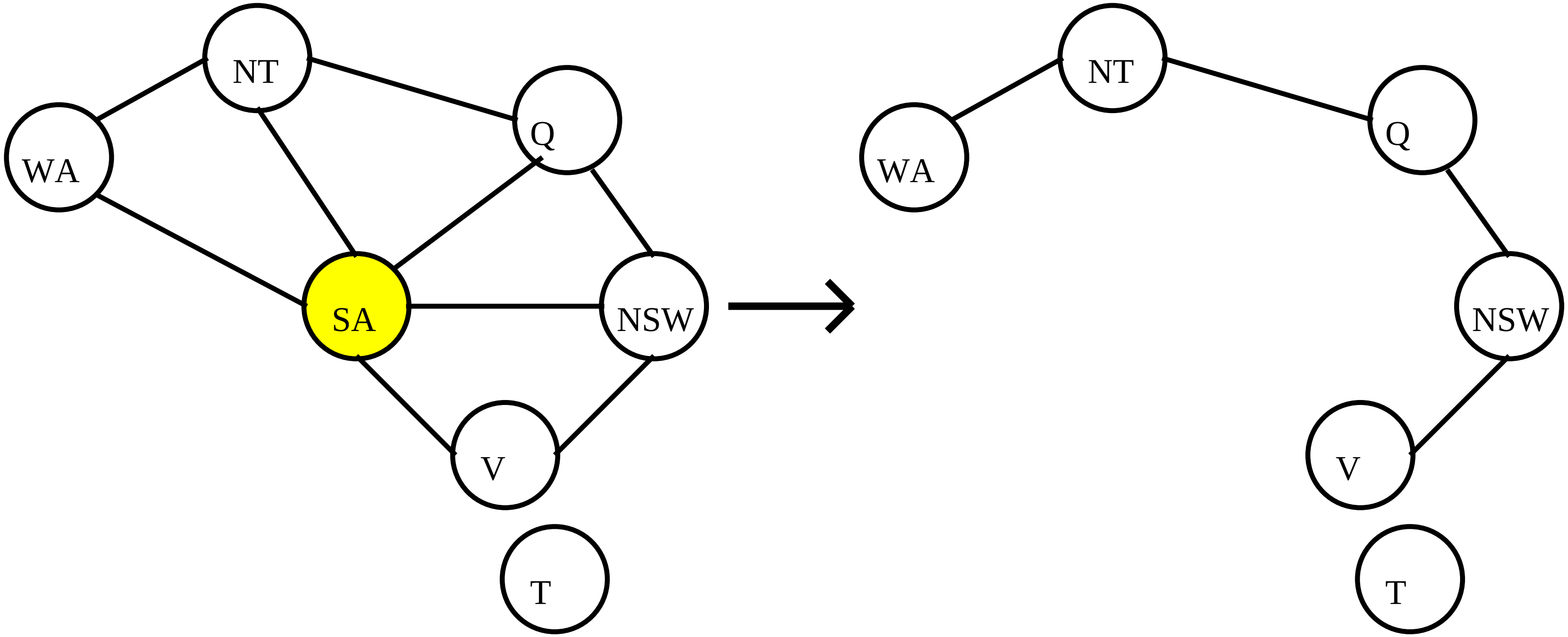
1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



2. For j from n down to 2, apply $\textsc{RemoveInconsistent}(Parent(X_j), X_j)$
3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Nearly tree-structured CSPs

Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

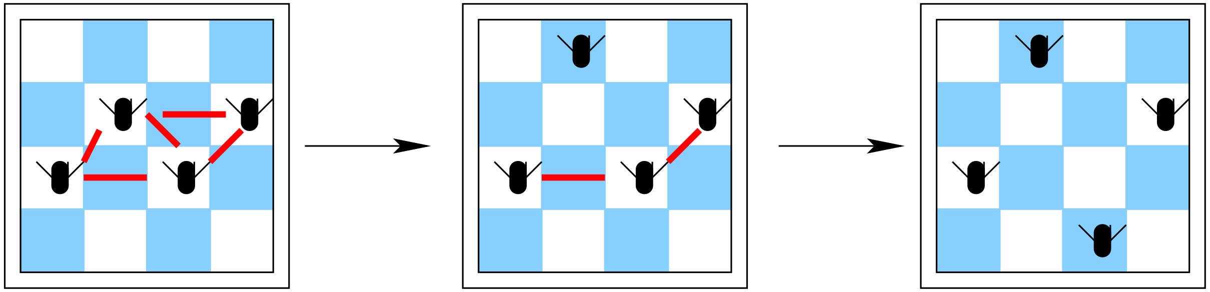
Cutset size $c \implies$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Iterative algorithms for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  allow states with unsatisfied constraints
  operators *reassign* variable values
- Variable selection: randomly select any conflicted variable
- Value selection by *min-conflicts* heuristic:
  choose value that violates the fewest constraints
  i.e., hillclimb with $h(n)$ = total number of violated constraints

Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)
Operators: move queen in column
Goal test: no attacks
Evaluation: $h(n)$ = number of attacks



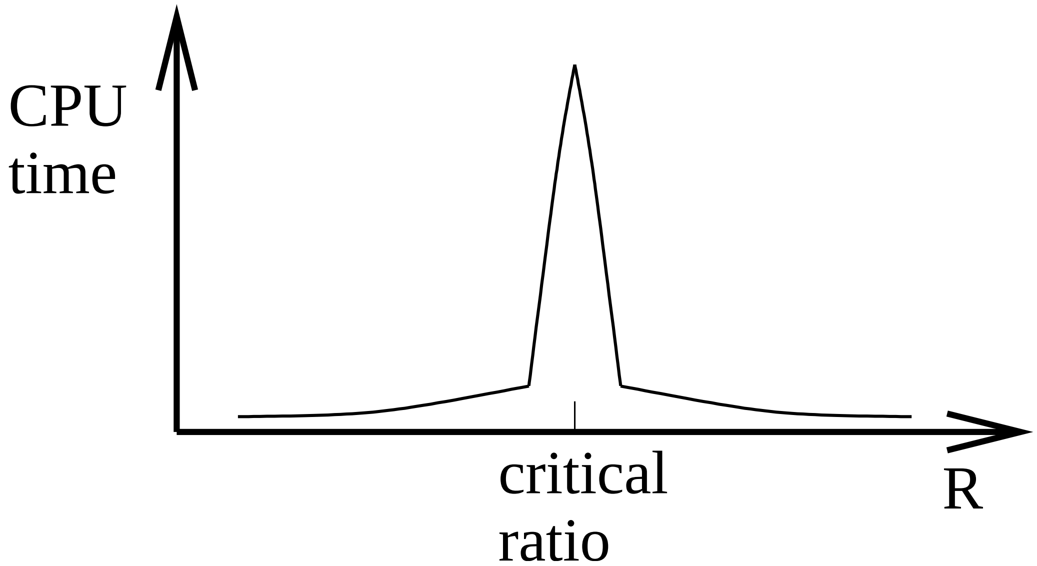
h=5 h=2 h=0

Performance of min-conflicts

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

- CSPs are a special kind of problem: states defined by values of a fixed set of variables goal test defined by *constraints* on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure

Summary

- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The CSP representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice