Propositional logic: Syntax

Propositional logic is the simplest logic—illustrates basic ideas

The proposition symbols $P_1$, $P_2$ etc are sentences

If $S$ is a sentence, $\neg S$ is a sentence (negation)

If $S_1$ and $S_2$ are sentences, $S_1 \land S_2$ is a sentence (conjunction)

If $S_1$ and $S_2$ are sentences, $S_1 \lor S_2$ is a sentence (disjunction)

If $S_1$ and $S_2$ are sentences, $S_1 \implies S_2$ is a sentence (implication)

If $S_1$ and $S_2$ are sentences, $S_1 \iff S_2$ is a sentence (biconditional)
Propositional logic: Semantics

Each model specifies true/false for each proposition symbol

E.g. $P_{1,2}$ $P_{2,2}$ $P_{3,1}$

true true false

(With these symbols, 8 possible models, can be enumerated automatically.)

Rules for evaluating truth with respect to a model $m$:

$\neg S$ is true iff $S$ is false

$S_1 \land S_2$ is true iff $S_1$ is true and $S_2$ is true

$S_1 \lor S_2$ is true iff $S_1$ is true or $S_2$ is true

$S_1 \Rightarrow S_2$ is true iff $S_1$ is false or $S_2$ is true

i.e., is false iff $S_1$ is true and $S_2$ is false

$S_1 \Leftrightarrow S_2$ is true iff $S_1 \Rightarrow S_2$ is true and $S_2 \Rightarrow S_1$ is true

Simple recursive process evaluates an arbitrary sentence, e.g.,

$\neg P_{1,2} \land (P_{2,2} \lor P_{3,1}) = true \land (false \lor true) = true \land true = true$
## Truth tables for connectives

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<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\neg P$</th>
<th>$P \land Q$</th>
<th>$P \lor Q$</th>
<th>$P \Rightarrow Q$</th>
<th>$P \equiv Q$</th>
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Wumpus world sentences

Let $P_{i,j}$ be true if there is a pit in $[i, j]$.
Let $B_{i,j}$ be true if there is a breeze in $[i, j]$.

$\neg P_{1,1}$
$\neg B_{1,1}$
$B_{2,1}$

“Pits cause breezes in adjacent squares”
Let $P_{i,j}$ be true if there is a pit in $[i, j]$.
Let $B_{i,j}$ be true if there is a breeze in $[i, j]$.

$\neg P_{1,1}$
$\neg B_{1,1}$
$B_{2,1}$

“Pits cause breezes in adjacent squares”

$B_{1,1} \iff (P_{1,2} \lor P_{2,1})$
$B_{2,1} \iff (P_{1,1} \lor P_{2,2} \lor P_{3,1})$

“A square is breezy if and only if there is an adjacent pit”
Truth tables for inference

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<th>$B_{1,1}$</th>
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<th>$R_1$</th>
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Enumerate rows (different assignments to symbols),
if KB is true in row, check that $\alpha$ is too
Inference by enumeration

Depth-first enumeration of all models is sound and complete

function TT-Entails?(KB, α) returns true or false
inputs: KB, the knowledge base, a sentence in propositional logic
        α, the query, a sentence in propositional logic
symbols ← a list of the proposition symbols in KB and α
return TT-Check-All(KB, α, symbols, [])

function TT-Check-All(KB, α, symbols, model) returns true or false
if EMPTY?(symbols) then
    if PL-True?(KB, model) then return PL-True?(α, model)
    else return true
else do
    P ← FIRST(symbols); rest ← REST(symbols)
    return TT-Check-All(KB, α, rest, EXTEND(P, true, model)) and
    TT-Check-All(KB, α, rest, EXTEND(P, false, model))

O(2^n) for n symbols; problem is co-NP-complete
Logical equivalence

Two sentences are logically equivalent iff true in same models:
\( \alpha \equiv \beta \) if and only if \( \alpha \models \beta \) and \( \beta \models \alpha \)

\[
\begin{align*}
(\alpha \land \beta) & \equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) & \equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) & \equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) & \equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg(\neg\alpha) & \equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) & \equiv (\neg \beta \Rightarrow \neg \alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) & \equiv (\neg \alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \iff \beta) & \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \land \beta) & \equiv (\neg \alpha \lor \neg \beta) \quad \text{De Morgan} \\
\neg(\alpha \lor \beta) & \equiv (\neg \alpha \land \neg \beta) \quad \text{De Morgan} \\
(\alpha \land (\beta \lor \gamma)) & \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) & \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\end{align*}
\]
Validity and satisfiability

A sentence is valid if it is true in all models,
e.g., True, A ∨ ¬A, A ⇒ A, (A ∧ (A ⇒ B)) ⇒ B

Validity is connected to inference via the Deduction Theorem:
KB ⊨ α if and only if (KB ⇒ α) is valid

A sentence is satisfiable if it is true in some model
e.g., A ∨ B, C

A sentence is unsatisfiable if it is true in no models
e.g., A ∧ ¬A

Satisfiability is connected to inference via the following:
KB ⊨ α if and only if (KB ∧ ¬α) is unsatisfiable
i.e., prove α by reductio ad absurdum
Proof methods

Proof methods divide into (roughly) two kinds:

**Application of inference rules**
- Legitimate (sound) generation of new sentences from old
- **Proof** = a sequence of inference rule applications
  - Can use inference rules as operators in a standard search alg.
  - Typically require translation of sentences into a normal form

**Model checking**
- truth table enumeration (always exponential in $n$)
- improved backtracking, e.g., Davis–Putnam–Logemann–Loveland
- heuristic search in model space (sound but incomplete)
  - e.g., min-conflicts-like hill-climbing algorithms
Forward and backward chaining

**Horn Form** (restricted)

KB = conjunction of Horn clauses

Horn clause =
- ♦ proposition symbol; or
- ♦ (conjunction of symbols) ⇒ symbol

E.g., \( C \land (B \Rightarrow A) \land (C \land D \Rightarrow B) \)

**Modus Ponens** (for Horn Form): complete for Horn KBs

\[
\frac{\alpha_1, \ldots, \alpha_n, \alpha_1 \land \cdots \land \alpha_n \Rightarrow \beta}{\beta}
\]

Can be used with forward chaining or backward chaining.
These algorithms are very natural and run in **linear** time
Forward chaining

Idea: fire any rule whose premises are satisfied in the $KB$, add its conclusion to the $KB$, until query is found

\[ P \Rightarrow Q \]
\[ L \land M \Rightarrow P \]
\[ B \land L \Rightarrow M \]
\[ A \land P \Rightarrow L \]
\[ A \land B \Rightarrow L \]
\[ A \]
\[ B \]
Forward chaining algorithm

**function** PL-FC-ENTAILS?\((KB, q)\) **returns** true or false

**inputs:** \(KB\), the knowledge base, a set of propositional Horn clauses
\(q\), the query, a proposition symbol

**local variables:** count, a table, indexed by clause, initially the number of premises
inferred, a table, indexed by symbol, each entry initially false
agenda, a list of symbols, initially the symbols known in \(KB\)

while agenda is not empty do
\(p \leftarrow \text{POP}(\text{agenda})\)
unless inferred\([p]\) do
\(\text{inferred}[p] \leftarrow \text{true}\)
for each Horn clause \(c\) in whose premise \(p\) appears do
\(\text{decrement} \ \text{count}[c]\)
if count\([c]\) = 0 then do
\(\text{if HEAD}[c] = q \text{ then return true}\)
\(\text{Push} \ (\text{HEAD}[c], \text{agenda})\)

return false
Forward chaining example
Forward chaining example
Forward chaining example
Forward chaining example
Forward chaining example
Forward chaining example
Forward chaining example
Forward chaining example
Proof of completeness

FC derives every atomic sentence that is entailed by $KB$

1. FC reaches a fixed point where no new atomic sentences are derived

2. Consider the final state as a model $m$, assigning true/false to symbols

3. Every clause in the original $KB$ is true in $m$
   **Proof**: Suppose a clause $a_1 \land \ldots \land a_k \Rightarrow b$ is false in $m$
   Then $a_1 \land \ldots \land a_k$ is true in $m$ and $b$ is false in $m$
   Therefore the algorithm has not reached a fixed point!

4. Hence $m$ is a model of $KB$

5. If $KB \models q$, $q$ is true in every model of $KB$, including $m$

**General idea**: construct any model of $KB$ by sound inference, check $\alpha$
Backward chaining

Idea: work backwards from the query $q$:
   to prove $q$ by BC,
       check if $q$ is known already, or
   prove by BC all premises of some rule concluding $q$

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal
   1) has already been proved true, or
   2) has already failed
Backward chaining example
Backward chaining example
Backward chaining example

![Diagram of backward chaining example with nodes labeled M, L, Q, P, A, and B connected by directed edges.]

Chapter 7 57
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Forward vs. backward chaining

FC is data-driven, cf. automatic, unconscious processing, e.g., object recognition, routine decisions

May do lots of work that is irrelevant to the goal

BC is goal-driven, appropriate for problem-solving, e.g., Where are my keys? How do I get into a PhD program?

Complexity of BC can be much less than linear in size of KB
Resolution

Conjunctive Normal Form (CNF—universal)

\textit{conjunction of \underline{disjunctions of literals} clauses}

E.g., \((A \lor \neg B) \land (B \lor \neg C \lor \neg D)\)

Resolution inference rule (for CNF): complete for propositional logic

\[
\ell_1 \lor \cdots \lor \ell_k, \quad m_1 \lor \cdots \lor m_n
\]

\[
\ell_1 \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n
\]

where \(\ell_i\) and \(m_j\) are complementary literals. E.g.,

\[
P_{1,3} \lor P_{2,2}, \quad \neg P_{2,2}
\]

\[
P_{1,3}
\]

Resolution is sound and complete for propositional logic
Conversion to CNF

\[ B_{1,1} \iff (P_{1,2} \lor P_{2,1}) \]

1. Eliminate \( \iff \), replacing \( \alpha \iff \beta \) with \( (\alpha \implies \beta) \land (\beta \implies \alpha) \).

\[ (B_{1,1} \implies (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \implies B_{1,1}) \]

2. Eliminate \( \implies \), replacing \( \alpha \implies \beta \) with \( \neg \alpha \lor \beta \).

\[ (\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg(P_{1,2} \lor P_{2,1}) \lor B_{1,1}) \]

3. Move \( \neg \) inwards using de Morgan’s rules and double-negation:

\[ (\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1}) \]

4. Apply distributivity law (\( \lor \) over \( \land \)) and flatten:

\[ (\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1}) \]
Resolution algorithm

Proof by contradiction, i.e., show $KB \land \neg \alpha$ unsatisfiable

function $\text{PL-Resolution}(KB, \alpha)$ returns true or false

inputs: $KB$, the knowledge base, a sentence in propositional logic
$\alpha$, the query, a sentence in propositional logic

$clauses \leftarrow$ the set of clauses in the CNF representation of $KB \land \neg \alpha$
$new \leftarrow \{\}$

loop do
  for each $C_i, C_j$ in $clauses$ do
    $resolvents \leftarrow$ $\text{PL-Resolve}(C_i, C_j)$
    if $resolvents$ contains the empty clause then return true
    $new \leftarrow new \cup resolvents$
  
  if $new \subseteq clauses$ then return false
  $clauses \leftarrow clauses \cup new$
Resolution example

\[ KB = (B_{1,1} \iff (P_{1,2} \lor P_{2,1})) \land \neg B_{1,1} \quad \alpha = \neg P_{1,2} \]